# Periodic Solution of Predator-prey Model in Semi Ratio Dependent with Holling III Type Functional Response and Impulsive Effect

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**Abstract:** The paper studies the response of function with Holling III type and the dynamics of predator-prey system in semi-ratio dependent. Based on some mathematical analysis theories, the paper makes use of Mawhin's continuation theorem, which gives the existence theorem of periodic solution of the system. In the end, the paper gives a numerical simulation example, which gets the validity of the results.

## 1. Introduction

In recent years, predator-prey systems in semi-proportional dependent with functional response have attracted more and more researchers attention (see [1-3]) However, we have developed biological species that experience steady variation for a long time, which can experience the rapid changed in a short period in a regular time. Therefore, there are more studies including impulsive differential equations applied to biological problems. Kuno adopt here a different expression to incorporate this factor, replacing birth rate $b$ by the function $bx(t)/(N+x(t))$ [4]. In this section, we will account the corresponding predator-prey system with sparse effect and impulses effect,

$$\begin{cases} \left.\begin{array}{l} \dot{x}_1 = x_1\left(\dfrac{a(t)x_1}{N+x_1} - c(t) - b(t)x_1\right) - \dfrac{s(t){x_1}^2 x_2}{A(t)+{x_1}^2}, \\ \dot{x}_2 = x_2\left(d(t) - e(t)\dfrac{x_2}{x_1}\right), \end{array}\right\} t \neq t_k, \\ \Delta x_i(t_k) = I_{ik} x_i(t_k^-), \qquad t = t_k, k = 1,2,.... \end{cases} \tag{1}$$

Where,at time $t$, $x_i$ is density of population . $k \in Z^+$, $t_1 < t_2 < \cdots < t_k < \cdots$ are impulse points such that $\lim t_k = +\infty$ . Suppose function $I_k(\cdot): R \mapsto R$ is continuous. Just like the general theory of impulsive differential equations [5], we Suppose that $x_i(t_k) = x_i({t_k}^-)$ at the points $t_k$ of the solution $t \mapsto x_i(t)$, here $t_k$ is discontinuous point. Suppose that

$(H_1)$ for all $t \in (0,+\infty)$, $e(t)$, $s(t), a(t), b(t), c(t), d(t)$ and $A(t)$ are strictly positive periodic functions with $\omega > 0$ ,And these functions are also continuous 、bounded .

$(H_2)$ Exists in a positive integer $q$ such that $I_{i(k+q)} = I_{ik}$ , $t_{k+q} = t_k + \omega$ . We suppose that $[0,\omega] \cap \{t_k\} = \{t_1, t_2, \cdots, t_m\}$ and $t_k \neq 0$, here $m = q$ .

$(H_3)$ for $1 + I_{ik} > 0$, there exists constants $I_i \geq 0$ such that $I_{ik} \leq I_i$ ,( $k = 1,2\cdots, m$ , $i = 1,2$ ).

## 2. Existence of Periodic Solutions

In this section, we use Mawhin's continuation theorem to prove the existence of periodic solutions of system (1). More details can be referred to [6].

**Lemma1.**([6]) Assume that

(a) every solution $x \in \partial\Omega \bigcap DomL$ for $\lambda \in (0,1)$ , $Lx \neq \lambda Nx$;

(b) $QNx$ is not zero for each $x \in \partial\Omega \bigcap KerL$;

(c) the *brouwer degree* $\deg\{JQN, \Omega \bigcap KerL, 0\}$ is not equal to zero.

where, $X$ and $Y$ be two Banach spaces, $L: DomL \bigcap X \to Y$ is a Fredholm mapping of index zero , Think about an operator equation $Lx = \lambda Nx$, here $\lambda \in [0,1]$ is a parameter. And suppose $N: \overline{\Omega} \to Y$ is $L$- compact on $\overline{\Omega}$, where $\Omega$ is open bounded in $X$ . Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap DomL$ .

**Theorem 1.** If $(H_1)-(H_3)$ hold, there is at least one positive periodic solution to equation (1).

$(H_4)$ $\bar{d}\omega + m\ln(1+I_2) > 0$, $\dfrac{\bar{c}N}{\bar{a}-\bar{b}} - \dfrac{N}{(\bar{a}-\bar{b})\omega}\sum_{k=1}^{m} B_{1k} > 0$

**Proof.** We can let

$$x_i = \exp\{y_i\}, i = 1,2 \tag{2}$$

then the equation (1) becomes

$$\begin{cases} \left.\begin{aligned} & y_1' = \left(\dfrac{a(t)}{N+\exp\{y_1\}} - b(t)\right)\exp\{y_1\} - c(t) - s(t)\dfrac{\exp\{y_1+y_2\}}{A(t)+\exp\{2y_1\}} \\ & y_2' = d(t) - \dfrac{e(t)\exp\{y_2\}}{\exp\{y_1\}} \end{aligned}\right\} t \neq t_k, \\ \Delta y_i(t_k) = \ln\{1+I_{ik}\} := B_{ik}, t = t_k. \end{cases} \tag{3}$$

Now, we need to show that there exists a domain $\Omega$ that satisfies all the requirements given in Lemma 1. we take

Set $L: DomL \bigcap Y \to Z$ , $Ly = (y', \Delta y(t_1), \cdots, \Delta y(t_m))$ , where $DomL = \{y \in C^1[0,T; t_1, \cdots, t_m]\}$ and $N: Y \to Z$ . Then $Y$ and $Z$ are Banach spaces when they are endowed with the norms $\|x\|_c = \sup\limits_{t\in[0,\omega]} |x(t)|$ and $\|(x, c_1, \ldots, c_m)\| = (\|x\|_c^2 + |c_1|^2 + \cdots + |c_m|^2)^{1/2}$ and take $Ly = y'$ , $Py = \dfrac{1}{\omega}\int_0^\omega y(t)dt, y \in Y$ , $Qz = Q(f, C_1, \cdots, C_m) = \left(\dfrac{1}{\omega}\left(\sum_{k=1}^{m} C_k + \int_0^\omega f(u)du\right), 0, \cdots, 0\right)$ . Let $\bar{u} = \dfrac{1}{\omega}\int_0^\omega u(t)dt$.

Then it's easy to prove $L$ is a Fredholm mapping of index zero. $P: Y \to KerL$ , $Q: Z \to Z/\operatorname{Im}L$ denotes projectors, and that $N$ is $L$-compact on $\overline{\Omega}$ for any given open and bound subset $\Omega$ in $X$ .

According to the equation $Ly = \lambda Ny$, $\lambda \in (0,1)$ , assume that $y(t) = (y_1, y_2) \in Y$ denotes a solution of system (4) for a certain $\lambda \in (0,1)$. By integrating (3) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega \frac{a(t)\exp\{y_1\}}{N+\exp\{y_1\}}dt = \int_0^\omega \left[c(t) + b(t)\exp\{y_1\} + \frac{s(t)\exp\{y_1+y_2\}}{A(t)+\exp\{2y_1\}}\right]dt - \sum_{k=1}^{m} B_{1k} \tag{4}$$

$$\bar{d}\omega = \int_0^\omega e(t)\frac{\exp\{y_2\}}{\exp\{y_1\}}dt - \sum_{k=1}^{m} B_{2k} \tag{5}$$

From (3),(4),(5), we obtain

$$\int_0^\omega |y_1'|dt < \int_0^\omega \frac{a(t)\exp\{y_1\}}{N+\exp\{y_1\}}dt + \int_0^\omega \left[ c(t) + b(t)\exp\{y_1\} + \frac{s(t)\exp\{y_1+y_2\}}{A(t)+\exp\{2y_1\}} \right]dt + \sum_{k=1}^m B_{1k} \tag{6}$$
$$\le 2\bar{a}\omega + 2m\ln(1+I_1) := 2M_1$$

$$\int_0^\omega |y_2'|dt < \int_0^\omega e(t)\frac{\exp\{y_2\}}{\exp\{y_1\}}dt + \int_0^\omega d(t)dt + \sum_{k=1}^m B_{2k} \le 2\bar{d}\omega + 2m\ln(1+I_2) := 2M_2 \tag{7}$$

Let that $(y_1(t), y_2(t)) \in Y$ , there exists $\xi_i, \eta_i \in [0,\omega]$ such that $y_i(\xi_i) = \inf_{t\in[0,\omega]} y_i(t)$ , $y_i(\eta_i) = \sup_{t\in[0,\omega]} y_i(t)$. By (4), (5) and (8), we have

$$\bar{a}\omega \ge \int_0^\omega c(t)dt + \int_0^\omega b(t)\exp\{y_1(\xi_1)\}dt - \sum_{k=1}^m B_{1k} \ge \bar{c}\omega - m\ln(1+I_1) + \bar{b}\omega\exp\{y_1(\xi_1)\},$$

$$\bar{d}\omega \ge \int_0^\omega e(t)\frac{\exp\{y_2(\xi_2)\}}{\exp\{y_1(\eta_1)\}}dt - \sum_{k=1}^m B_{2k} \ge \bar{e}\omega\frac{\exp\{y_2(\xi_2)\}}{\exp\{y_1(\eta_1)\}} - m\ln(1+I_2)\,.$$

That is

$$y_1(t) \le y_1(\xi_1) + \sum_{k=1}^m |B_{1k}| + \int_0^\omega |y_1'(t)|dt < L_1 + \sum_{k=1}^m |B_{1k}| + 2M_1 =: H_1 \tag{8}$$

$$y_2(t) \le y_2(\xi_2) + \sum_{k=1}^m |B_{2k}| + \int_0^\omega |y_2'(t)|dt < \ln\left\{\frac{M_2}{\bar{e}\omega}\right\} + y_1(\eta_1) + \sum_{k=1}^m |B_{2k}| + 2M_2$$
$$< \ln\left\{\frac{M_2}{\bar{e}\omega}\right\} + H_1 + \sum_{k=1}^m |B_{2k}| + 2M_2 := H_2$$

$$y_1(\xi_1) \le \ln\frac{1}{\bar{b}}\left\{(\bar{a}-\bar{c}) + \frac{2m}{\omega}\ln(1+I_1)\right\} =: L_1, \quad y_2(\xi_2) \le \ln\left\{\frac{M_2}{\bar{e}\omega}\right\} + y_1(\eta_1) =: L_2 \tag{9}$$

By (4), also have

$$\int_0^\omega a(t)\frac{\exp\{y_1\}}{N}dt \ge \int_0^\omega \frac{a(t)\exp\{y_1\}}{N+\exp\{y_1\}}dt \ge \bar{c}\omega + \int_0^\omega [b(t)\exp\{y_1(t)\}]dt - \sum_{k=1}^m B_{1k}$$
$$\left(\frac{\bar{a}}{N} - \bar{b}\right)\omega\exp\{y_1(\eta_1)\} \ge \bar{c}\omega - \sum_{k=1}^m B_{1k};$$

which implies

$$y_2(\eta_1) \ge \ln\left\{\frac{\bar{c}N}{\bar{a}-\bar{b}} - \frac{N}{(\bar{a}-\bar{b})\omega}\sum_{k=1}^m B_{1k}\right\} =: l_1.$$

$$y_1(t) \ge l_1 - \sum_{k=1}^m |B_{1k}| - \int_0^\omega |y_1'(t)|dt = l_1 - \sum_{k=1}^m |B_{1k}| - 2M_1 =: H_3 \tag{10}$$

By (5), we also have

$$\int_0^\omega e(t)\frac{\exp\{y_2(\eta_2)\}}{\exp\{y_1(\xi_1)\}}dt \ge \bar{d}\omega + \sum_{k=1}^m B_{2k},.$$

That is

$$y_2(\eta_2) \geq y_1(\xi_1) - \ln \bar{e} + \ln\left[\bar{d}\omega + m\ln(1+I_2)\right]$$
$$\geq l_1 - m\ln(1+I_1) - 2M_1 - \ln \bar{e} + \ln\left[\bar{d}\omega + m\ln(1+I_2)\right] =: l_2.$$

Then

$$y_2(t) \geq y_2(\eta_2) - \sum_{k=1}^{m}|B_{2k}| - \int_0^{\omega}|y_2'(t)|dt > l_2 - \sum_{k=1}^{m}|B_{2k}| - 2M_2 =: H_4 \tag{11}$$

Hence, (8), (9), (10) and (11) imply that $\sup_{t\in[0,\omega]}|y_1(t)| < \sup_{t\in[0,\omega]}\{|H_1|,|H_3|\} := D_1$, $\sup_{t\in[0,\omega]}|y_2(t)| < \sup_{t\in[0,\omega]}\{|H_2|,|H_4|\} := D_2$. Clearly, $D_i$, $i=1,2,3$ are independent of $\lambda$. Denote $D = D_1 + D_2 + D_3$, where $D_3 > 0$ is taken sufficiently large such that $\|(\ln x_1^*, \ln x_2^*)\| = |\ln x_1^*| + |\ln x_2^*| < D_3$ and define $\Omega = \{y(t) \in Y : \|x\| < D\}$. By now we have proved that $\Omega$ satisfies all the requirements of Lemma 1. Hence, we derive that system (1) has at least one positive $\omega$- periodic solution. The proof is complete.

## 3. An Example of Numerical Simulations

In this section, some numerical examples are given to verify the validity and correctness of the theoretical results. For system (1), we take

$$I_{1k} = 0.3,\quad a(t) = 2 + 0.3\sin t,\qquad b(t) = 1 + 0.5\cos t,\quad s(t) = 1 + 0.1\sin t,$$

$$I_{2k} = 0.2,\quad A(t) = 1.2 + 0.2\sin t,\quad d(t) = 3 + 0.1\sin t,\quad e(t) = 0.8 + 0.3\cos t.$$

Obviously, they satisfy $(H_1)$.

If $T = 2\pi/3$, then system (1) has a unique $2\pi$-periodic solution under the conditions $(H_4)$. (see Fig.1).

Due to the influence of periodic pulse, the influence of pulse is obvious.

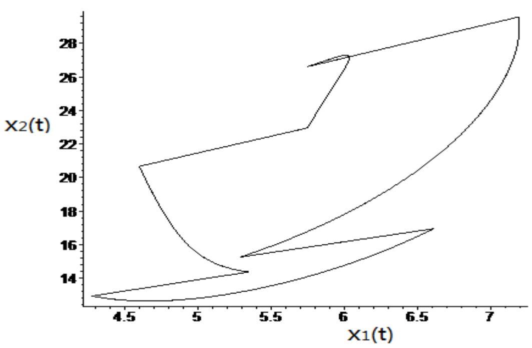


Fig.1 $T = 2\pi/3$ periodic solution

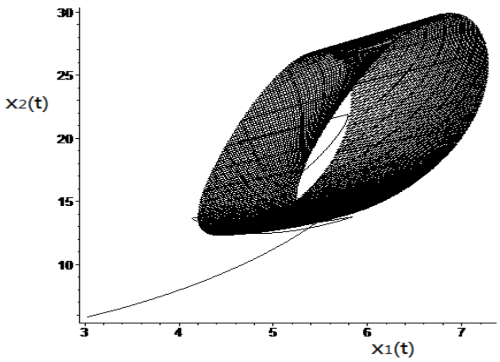


Fig.2 $T = 2$ Gui-Chaotic attractor

If $T = 2$, then $(H_3)$ isn't satisfied. Numeric results show that system (1) has still has a global attractor which can be a q chaotic strange attractor (see Fig.2). Every solutions of system (1) will finally tend to the quasi-periodic solution.

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## References

[1] Q. Wang, M. Fan, K. Wang. Dynamics of a class of nonautonomous semi-ratio-dependent predator-prey systems with functional responses. *J. Math. Anal. Appl*. 2003, 278:443-471.

[2] H.F. Huo. Periodic solutions for a semi-ratio-dependent predator–prey system with functional responses. Appl. Math. Lett., 18 (2005): 313–320.

[3] C.S. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomol. Soc. Canad*. 1965, 46: 1- 60.

[4] E. Kuno, Principles of Predator-Prey interaction in theoretical experimental and natural population systems. Advances in Ecological Research, 1987, 16 : 250-255 .

[5] W.P. Zhang, M. Fan. Periodic in a Generalized Ecological Competition System Governed by Impulsive Differential Equations with Delays. *Mathematical and Computer Modelling*. 2004, 39:479-493.

[6] R.E. Gaines, J.L. Mawhin. *Coincidence degree and nonlinear differential equations*. Springer-Verlag, Berlin, 1977.